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LINEAR ALGEBRA AND ITS APPLICATIONS

Derivation ranges

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Abstract

In this paper we obtain some sufficient and some necessary conditions that the identity be in the closure of the range of an inner derivation. We obtain some results concerning the intersection of the closure of the range of the inner derivation induced by A and the commutant of A.

1. Introduction

Let $\mathscr{L}(\mathscr{H})$ denote the algebra of all bounded linear operators on a complex separable and infinite dimensional Hilbert space $\mathscr{H}$. The inner derivation induced by $A \in \mathscr{L}(\mathscr{H})$ is the map defined by

$$\delta_A : \mathscr{L}(\mathscr{H}) \mapsto \mathscr{L}(\mathscr{H}) \qquad \delta_A(X) = AX - XA \ (A \in \mathscr{L}(\mathscr{H})).$$

The identity is not a commutator; that is, $I \notin R(\delta_A)$ for any $A \in \mathscr{L}(\mathscr{H})$, where $R(\delta_A)$ denotes the range of δ_A. Nevertheless, Anderson [1] proved the remarkable result that $I \in \overline{R(\delta_A)}$ for a large class of operators, where $\overline{R(\delta_A)}$ denotes the closure of the range of δ_A in the norm topology. This allowed him to define a new class of operators, called

$$J_A(\mathscr{H}) = \{A \in \mathscr{L}(\mathscr{H}):\ I \in \overline{R(\delta_A)}\}.$$

In this paper we obtain some sufficient conditions under which $I \notin \overline{R(\delta_A)}$ by generalizing some results given by Stampfli [2]. Let

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$$\mathcal{N} = \bigcup \{R(\delta_A) \cap \{A\}' \colon A \in \mathcal{L}(\mathcal{H})\},$$

where $\{A\}'$ denotes the commutant of A. In finite dimensions the set $\mathcal{N}$ is exactly the set of nilpotent operators. In infinite dimensions, the theorem of Kleinecke–Shirokov [3] shows that every operator in $\mathcal{N}$ is quasinilpotent. If we consider now instead of $\mathcal{N}$ the set

$$\mathcal{M} = \bigcup \left\{\overline{R(\delta_A)} \cap \{A\}' \colon A \in \mathcal{L}(\mathcal{H})\right\}$$

the theorem of Kleineck–Shirokov cannot be used. In other words, an operator in $\mathcal{M}$ is not necessarily quasinilpotent: we can take as a counter example the existence of an operator $A \in \mathcal{L}(\mathcal{H})$ such that $I \in \overline{R(\delta_A)}$. Anderson [4], pp. 135–136 proved that $\overline{R(\delta_A)} \cap \{A\}' = \{0\}$ if A is normal or isometric. Here we prove that every operator in $\mathcal{M}$ is nilpotent if $P(A)$ is normal, isometric or co-isometric for some polynomial P.

Weber [5] shows that every compact operator in $\overline{R(\delta_A)}^{w} \cap \{A\}'$ is quasinilpotent, where $\overline{R(\delta_A)}^{w}$ is the weak closure of $R(\delta_A)$. If we consider now the set

$$\left\{\overline{R(\delta_A)}^{w} \cap \{A^*\}' \colon A \in \mathcal{L}(\mathcal{H})\right\}$$

we can ask: is every compact operator in $\overline{R(\delta_A)}^{w} \cap \{A^*\}'$ quasinilpotent? We can partially answer this question by adding the assumption that A is dominant.

2. Derivation ranges and the identity

Definition 1. An isolated point λ_0 of $\sigma(A)$ (the spectrum of A), is called a pole of A of order v if $(\lambda I - A)^{-1}$ has a pole of order v at λ_0; this is equivalent [6], p. 573 to $(A - \lambda I)^v E_\lambda = 0$ and $(A - \lambda I)^{v-1} E_\lambda \neq 0$, where E_λ is the Riesz projection.

Lemma 1 ([2], p. 521). *Let $A \in \mathcal{L}(\mathcal{H})$. Then the following statements are equivalent.*

1. *$I \in \overline{R(\delta_A)}$*
2. *There exists an invertible operator $B \in \{A\}'$ such that $B \in \overline{R(\delta_A)}$*
3. *$\overline{R(\delta_A)}$ contains all the invertible operators in $\{A\}'$.*

Theorem 1. *Let $A \in \mathcal{L}(\mathcal{H})$ and suppose that there exists a non constant analytic function on an open set containing $\sigma(A)$ such that $f(A)$ is compact. Then $I \notin \overline{R(\delta_A)}$.*

Proof. If $f(A)$ is compact, then A is polynomially compact (see [7], p. 37), hence it suffices to prove that, if $P(A)$ is compact for some polynomial P, then $I \notin \overline{R(\delta_A)}$. Suppose that $I \in \overline{R(\delta_A)}$, let P be a polynomial of degree n for which $P(A)$ is compact and let $P^{(k)}$ denote the kth derivative of P. If

$$I \in \overline{R(\delta_A)},$$

then there exists a sequence (X_n) in $\mathscr{L}(\mathscr{H})$ such that

$$AX_n - X_nA \to I$$

since $I \in \{A\}'$ it easy to verify that

$$P^{(k)}(A)X_n - X_nP^{(k)}(A) \to P^{(k+1)}(A)I.$$

Then

$$P(A)X_n - X_nP(A) \to P^{(1)}(A),$$

which implies that $P^{(1)}(A)$ is compact. Also

$$P^{(1)}(A)X_n - X_nP^{(1)}(A) \to P^{(2)}(A)$$

implies that $P^{(2)}(A)$ is compact. By repeating the same argument it follows that $P^{(n)}(A)$ is compact, hence I is also compact, which is absurd. □

Remark 1. The above theorem generalizes the theorem of Stampfli [2], p. 522 (if A^k is compact, then $I \notin \overline{R(\delta_A)}$).

Theorem 2. *Let $A \in L(H)$ and suppose that*

$$\overline{R(\delta_{p(A)})} \cap \{P(A)\}' = \{0\}$$

for some polynomial P. Then every operator in $\overline{R(\delta_A)} \cap \{A\}'$ is nilpotent.

Proof. Suppose that $T \in \overline{R(\delta_A)} \cap \{A\}'$. Then there exists (X_n) in $L(H)$ such that

$$AX_n - X_nA \to T \in \{A\}'.$$

Then

$$P(A)X_n - X_nP(A) \to P^{(1)}(A)T$$

which shows that

$$P^{(1)}(A)T \in \overline{R(\delta_{P(A)})} \cap \{P(A)\}'$$

that is, $P^{(1)}(A)T = 0$. Now

$$P^{(1)}(A)X_n - X_nP^{(1)}(A) \to P^{(2)}(A)T$$

which gives

$$0 = TP^{(1)}(A)X_nT - TX_nP^{(1)}(A)T \to P^{(2)}(A)T^3$$

that is, $P^{(2)}(A)T^3 = 0$. By repeating the same argument it follows that $T^k = 0$ for a given integer k, hence T is nilpotent. In particular every normal operator in $\overline{R(\delta_A)} \cap \{A\}'$ vanishes. □

Theorem 3. *Let $A \in \mathscr{L}(\mathscr{H})$ and suppose that*

$$\overline{R(\delta_{P(A)})} \cap \{P(A)\}' = \{0\}$$

for some polynomial P. Then $I \notin \overline{R(\delta_A)}$.

Proof. Suppose that $I \in \overline{R(\delta_A)}$. By Lemma 1, there exists an invertible operator B such that

$$B \in \overline{R(\delta_A)} \cap \{A\}'.$$

But then Theorem 2 would imply that B is nilpotent, which is absurd. □

Corollary 1. *Let $A \in \mathscr{L}(\mathscr{H})$. If $P(A)$ is normal, isometric or co-isometric ($AA^* = I$ or $A^*A = I$) for some polynomial P, then $I \notin \overline{R(\delta_A)}$.*

Proof. In [4], p. 136–137, Anderson shows that

$$\overline{R(\delta_{P(A)})} \cap \{P(A)\}' = \{0\}$$

under these hypotheses. □

Lemma 2. *Let $A \in \mathscr{L}(\mathscr{H})$ and f be an analytic function on an open set containing $\sigma(A)$. If $T \in \overline{R(\delta_A)} \cap \{A\}'$, then $f'(A)T \in \overline{R(\delta_{f(A)})}$.*

Proof. We have

$$f(A) = \frac{1}{2\pi i}\int_\gamma f(\lambda)(\lambda - A)^{-1}\,\mathrm{d}\lambda,$$

$$f'(A) = \frac{1}{2\pi i}\int_\gamma f(\lambda)(\lambda - A)^{-2}\,\mathrm{d}\lambda.$$

Suppose that $T \in \overline{R(\delta_A)} \cap \{A\}'$. Then there exists a sequence (X_n) in $\mathscr{L}(\mathscr{H})$ such that

$$\|T - (AX_n - X_nA)\| \to 0.$$

We have

$$f(A)X_n - X_nf(A) - f'(A)T = \frac{1}{2\pi i}\int_{\gamma} f(\lambda)[(\lambda - A)^{-1}X_n - X_n(\lambda - A)^{-1} - (\lambda - A)^{-2}]\mathrm{d}\lambda$$

since $T \in \{A\}'$. Then

$$(\lambda - A)^{-2}T = (\lambda - A)^{-1}T(\lambda - A)^{-1}.$$

Consequently we obtain

$$\begin{aligned}&(\lambda - A)^{-1}X_n - X_n(\lambda - A)^{-1} - (\lambda - A)^{-2}T\\ &\quad= (\lambda - A)^{-1}[X_n(\lambda - A) - (\lambda - A)X_n - T](\lambda - A)^{-1}\\ &\quad= (\lambda - A)^{-1}(AX_n - X_nA - T)(\lambda - A)^{-1}.\end{aligned}$$

Hence

$$\left\|f(\lambda)[(\lambda - A)^{-1}X_n - X_n(\lambda - A)^{-1} - (\lambda - A)^{-2}]\right\| \leqslant \sup_{\lambda\in\gamma}\|f(\lambda)\|\|(\lambda - A)^{-1}\|^2\|AX_n - X_nA - T\|$$

which implies that

$$\|f(A)X_n - X_nf(A) - f'(A)T\| \to 0$$

i.e., $f'(A)T \in \overline{R(\delta_{f(A)})}$. $\square$

Theorem 4. *Let $A \in \mathscr{L}(\mathscr{H})$ and f be an analytic function on an open set containing $\sigma(A)$ such that f' does not vanish on $\sigma(A)$. Then*

$$A \in J_A(\mathscr{H}) \Longleftrightarrow f(A) \in J_A(\mathscr{H})$$

Proof. $\Rightarrow$) Suppose that $I \in \overline{R(\delta_A)}$ $(0A \in J_A(\mathscr{H}))$. Then by Lemma 2, we have

$$f'(A) \in \overline{R(\delta_{f(A)})} \cap \{f(A)\}'.$$

Since $f'(A)$ is invertible, it follows by Lemma 1, that $I \in \overline{R(\delta_{f(A)})}$.

$\Leftarrow$) The result of Weber [8] guarantees that, if f is an analytic function on an open set containing $\sigma(A)$, we have $R(\delta_{f(A)}) \subset R(\delta_A)$, so if $I \in \overline{R(\delta_{f(A)})}$, then $I \in \overline{R(\delta_A)}$. $\square$

Lemma 3. *Let $A \in \mathscr{L}(\mathscr{H})$ be an operator which has a pole of finite order. Then*

$$I \notin \overline{R(\delta_A)}.$$

Proof. If the pole has order v, in the decomposition of $\mathscr{H} = R(E_\lambda) \oplus G$, we can write $A = A_1 \oplus A_2$, where $(A_1 - \lambda)^v = 0$ on $R(E_\lambda)$. Since A_1 is an algebraic operator, $I \notin \overline{R(\delta_{A_1})}$. Hence

$$I \notin \overline{R(\delta_A)}. \quad \square$$

Theorem 5 [9]. *Let $A \in \mathscr{L}(\mathscr{H})$ and suppose that f is an analytic function on an open set containing $\sigma(A)$ such that f' does not vanish on some neighbourhood of $\sigma(A)$. If*

$$\overline{R(\delta_{f(A)})} \cap \{f(A)\}' = \{0\},$$

then $I \notin \overline{R(\delta_{f(A)})}$.

Remark 2. In the proof of Theorem 5 it was asserted by Yang [9] that if $I \in \overline{R(\delta_A)}$, then $\sigma(A)$ is finite, hence the operator A is similar to an operator of the form $\sum_{i=1}^{n} \oplus A_i$ with $\sigma(A_i) = \{\lambda_i\}$ and $A_i - \lambda_i$ is nilpotent. This is not true in general.

Thus we provide the following proof.

Proof. It follows from Lemma 3 that we can assume A has no poles. Suppose that $I \in \overline{R(\delta_A)}$. Then there exists a sequence (X_n) of operators in $\mathscr{L}(\mathscr{H})$ such that

$$AX_n - X_n A \to I,$$

hence by Lemma 2

$$f(A)X_n - X_n f(A) \to f'(A).$$

Therefore $f'(A) \in \overline{R(\delta_{f(A)})} \cap \{f(A)\}' = \{0\}$ implies $f'(A) = 0$ and by the theorem of the minimal equation [6], p. 571, f' vanishes on some neighborhood of $\sigma(A)$. Hence $I \notin \overline{R(\delta_A)}$. $\square$

Corollary 3. *Let $A \in \mathscr{L}(\mathscr{H})$ and let $f(A)$ be normal, isometric or co-isometric, where f is an analytic function on an open set containing $\sigma(A)$ such that f' does not vanish on some neighborhood of $\sigma(A)$. Then $I \notin \overline{R(\delta_A)}$.*

Proof. Since $f(A)$ is normal, isometric or co-isometric

$$\overline{R(\delta_{f(A)})} \cap \{f(A)\}' = \{0\}.$$

Hence by Theorem 6 $I \notin \overline{R(\delta_A)}$. $\square$

Remark 3. In Corollary 3, the hypothesis that f' not vanish on some neighborhood of $\sigma(A)$ is indispensable. Indeed, there exists an operator $A \in \mathscr{L}(\mathscr{H})$ such that $I \in \overline{R(\delta_A)}$; nevertheless, $f(A)$ is obviously normal for any function f vanishing on some neighborhood of $\sigma(A)$. Therefore, the theorem of

Stampfli [2] p. 522 is true if we assume that f' does not vanish on some neighborhood of $\sigma(A)$.

3. Commutants and derivation ranges

Definition 2. An operator $A \in \mathscr{L}(\mathscr{H})$ is called, by Stampfli and Wadhwa [10], dominant if, for all complex λ, range $(A-\lambda) \subseteq \text{range}(A-\lambda)^*$, or equivalently, if there is a real number $M_\lambda \geqslant 1$ such that

$$\|(A-\lambda)^* f\| \leqslant M_\lambda \|(A-\lambda)f\|$$

for all f in $\mathscr{H}$. If there is a constant M such that $M_\lambda \leqslant M$ for all λ, A is called M-hyponormal, and if $M = 1$, A is hyponormal.

Theorem 6 [5]. *Let $A \in \mathscr{L}(\mathscr{H})$. Then every compact operator in $\overline{R(\delta_A)}^{w} \cap \{A\}'$ is quasinilpotent.*

Theorem 7. *If $B \in \overline{R(\delta_A)}^{w} \cap \{A\}'$ and $f(B)$ is compact, where f is an analytic function on an open set containing $\sigma(A)$, then*

$$\sigma(B) \subset \{z\colon zf(z) = 0\}.$$

Proof. If $B \in \overline{R(\delta_A)}^{w} \cap \{A\}'$, then

$$AX_\alpha - X_\alpha A \xrightarrow{w} B.$$

Since $f(B) \in \{A\}'$ we have

$$AX_\alpha f(B) - X_\alpha f(B)A \xrightarrow{w} Bf(B).$$

That is,

$$Bf(B) \in \overline{R(\delta_A)}^{w} \cap \{A\}'.$$

Since $Bf(B)$ is compact, then $\sigma(Bf(B)) = g(\sigma(B)) = 0$ by Theorem 6, where $g(z) = zf(z)$. $\square$

Theorem 8. *Let A* (resp. A^*) *be dominant operators. If $B \in \overline{R(\delta_A)}^{w} \cap \{A^*\}'$, then*

$$\{\lambda \in \sigma_p(B^*)\colon \dim \ker(B^* - \bar{\lambda}) < \infty\} \subset \{0\}$$

(respectively

$$\lambda \in \sigma_p(B)\colon \dim \ker(B - \lambda) < \infty\} \subset \{0\}).$$

where $\sigma_p(A)$ is the point spectrum of A.

Proof. Suppose that A is dominant and $B \in \overline{R(\delta_A)}^{w} \cap \{A^*\}'$. Then

$$B^* \in \overline{R(\delta_A)}^w \cap \{A\}'.$$

Let $\lambda \in \sigma_p(B^*)$ be such that $E = \ker(B^* - \lambda)$ is finite dimensional. The subspace E is invariant under B^* and A. It's easy to verify that $A|_E$ is dominant. Then $A|_E$ is normal so E reduces A, since E is finite dimensional. (see [10]).

Let $H = E \oplus E^{\perp}$. Then we can write

$$A = \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix}, \qquad B^* = \begin{pmatrix} \lambda & * \\ 0 & * \end{pmatrix}.$$

Since $B^* \in \overline{R(\delta_A)}^w$, $\lambda I_E \in R(\delta_{C^*})$, and this implies that $\lambda = 0$. By the same arguments as above, the proof of the theorem can be finished. □

Corollary 4. *If A or A^* are dominant operators, then every compact operator in $\overline{R(\delta_A)}^w \cap \{A^*\}'$ is quasinilpotent.*

Proof. Suppose that $B \in \overline{R(\delta_A)}^w \cap \{A^*\}'$ with B compact and $\lambda \in \sigma(B) \setminus \{0\}$. Then $\lambda \in \sigma_p(B)$ with $\dim \ker(B - \lambda) < \infty$ and $\overline{\lambda} \in \sigma_p(B^*)$ with $\dim \ker(B^* - \overline{\lambda}) < \infty$. Thus B is quasinilpotent by Theorem 8. □

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